SILAGE PRESSURES IN TOWER SILOS. PART 1. THEORETICAL AND DESIGN CONSIDERATIONS

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An analysis of the active and passive pressure fields developed by silage materials in cylindrical tower silos is presented in this study. The solutions are obtained in closed form and consider the variation of silage density with depth of fill, internal friction of silage, wall friction, diameter and height of the silo. The practical application of the results in predicting silo wall loadings in light of the unloader systems is discussed. Finally, the implications for the structural design of tower silos are examined.

INTRODUCTION

The estimation of pressures exerted by silage materials on the containing silo structure continues to be a critical task for design engineers (Mohsenin 1970). The available design formulas are empirical and take no account of the physical properties of the contained material.

In allied fields, the Janssen theory (Janssen 1895) is most widely used by silo designers. In his theory, Janssen made two assumptions: first, that stresses are independent of the horizontal coordinates and, second, that the ratio of lateral to vertical pressures, *K*, is a constant relative to depth of fill. The apparent weakness of predictions based on the Janssen equation lies mainly in the selection of a proper value for the pressure-ratio *K*.

The bulk density of the stored material is also assumed constant in the Janssen formula. Thus the validity of the Janssen equation for compressible materials like silage is questionable because experimental observations (Otis and Pomroy 1957; Aldrich 1963) indicate that silage density varies significantly with the depth of storage.

It is the purpose of this study to consider the pertinent parameters and set out a rational procedure for determining pressures developed by silage materials in cylindrical tower silos.

NOMENCLATURE

a = parameter: variation of unit weight
b = parameter: distribution of vertical pressure
c = subscript: conditions at silo axis
D = diameter of silo
f = frictional stress
M = moisture content
n = parameter: n = 1 for active case, n = -1 for passive case
p = lateral pressure
q = vertical pressure
R = radius of silo
r = radial coordinate
s = circumferential pressure
w = subscript: conditions at silo wall
z = depth coordinate
β = coefficient: unit weight
γ = unit weight
γ_o = initial unit weight
δ = wall friction angle

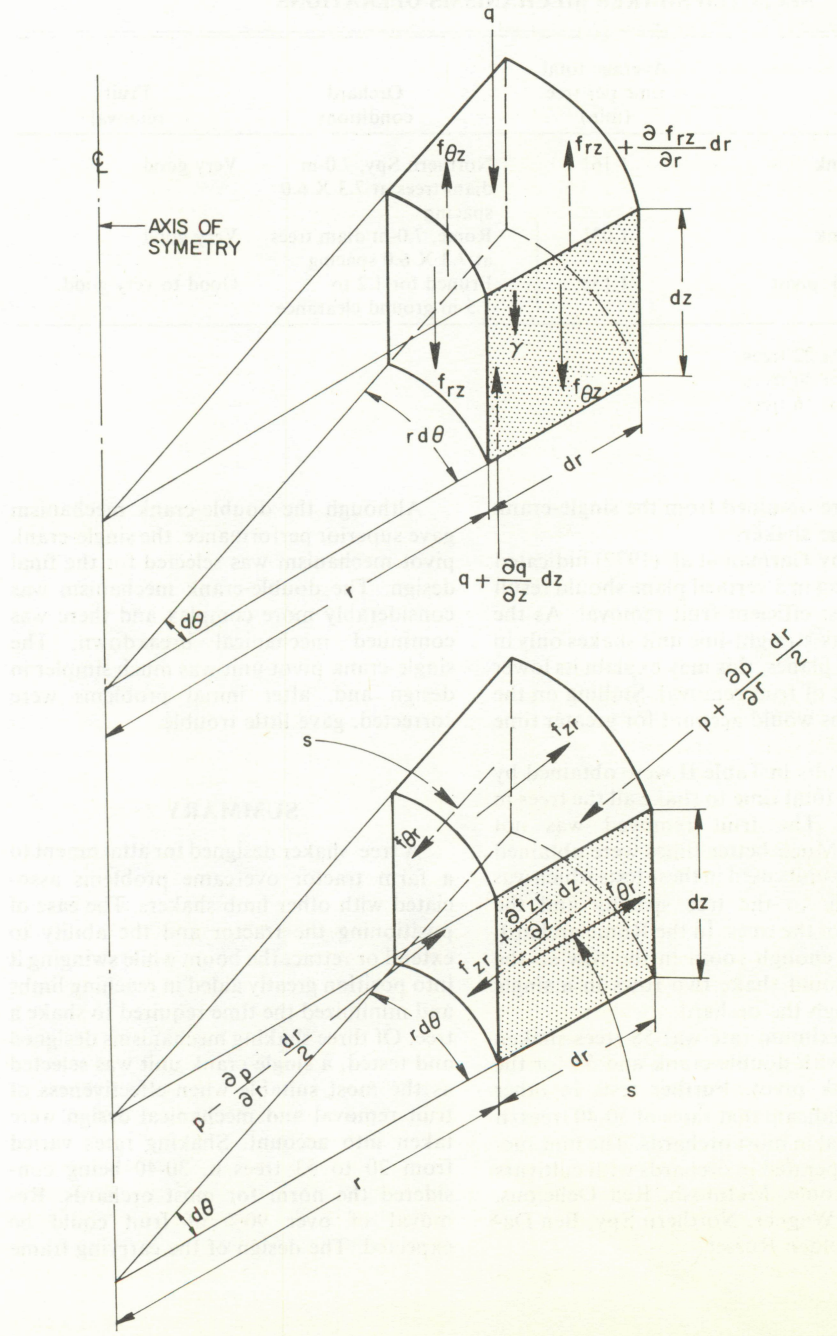

Figure 1. Variation of pressure on a volume element in *r* and *z* directions.

θ = circumferential coordinate
ϕ = effective angle of internal friction
ψ = angle between major principal pressure axis and vertical axis.

TERMINOLOGY

Definitions of terms as given by Jenike et al. (1973) are: "Active pressure field — field in which the major pressure is vertical and/or close to vertical. Passive pressure field — field in which the major pressure is horizontal and/or close to horizontal. Mass flow — flow pattern in which all solid in a bin is in motion whenever any of it is drawn out. Funnel flow — flow pattern in which solid flows in a channel formed within stagnant solid."

This nomenclature and terminology will be used in the companion papers.

THEORETICAL CONSIDERATIONS

Pressure Equilibrium Equations

Consider a volume element of silage in a tower silo, bounded by two concentric cylindrical surfaces, two horizontal surfaces and two axial planes. Figure 1 shows the variation of pressures acting on the sides of the volume element that contribute to its equilibrium in the r and z directions as defined in cylindrical coordinates. The silage material is assumed to be compressible, frictional, cohesive, isotropic and treated as a continuum. The pressure components are considered to be independent of the circumferential coordinate θ. The unit weight of the silage material γ is assumed to be a function of the depth coordinate z alone.

By summing up the forces on the volume element in the z and r directions, making the small angle approximations and neglecting higher-order terms, the following equations of equilibrium are obtained.

$$\frac{\partial q}{\partial z} + \frac{\partial f}{\partial r} + \frac{f}{r} = \gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{\partial f}{\partial z} + \frac{\partial p}{\partial r} + \frac{p - s}{r} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

where p, q, s denote the lateral, vertical and circumferential pressure components, respectively, and f is the frictional stress.

Variation of Unit Weight of Silage with Depth

A great deal of experimental work on the variation of unit weight of silage materials in tower silos was done by Perkins et al. (1953) Otis and Pomroy (1957) Shepherd and Woodward (1941) and Eckles et al. (1919) and culminated in the work of Aldrich (1963), who reduced the available data to a useful form. A suitable expression was developed by the present authors to describe the experimental curve obtained from the "unit weight — silage depth" data compiled by Aldrich.

The proposed equation is an exponential function of the form

$$\gamma(z) = \gamma_0 + a(1 - e^{-\beta z}) \quad \ldots\ldots\ldots\ldots \quad (3)$$

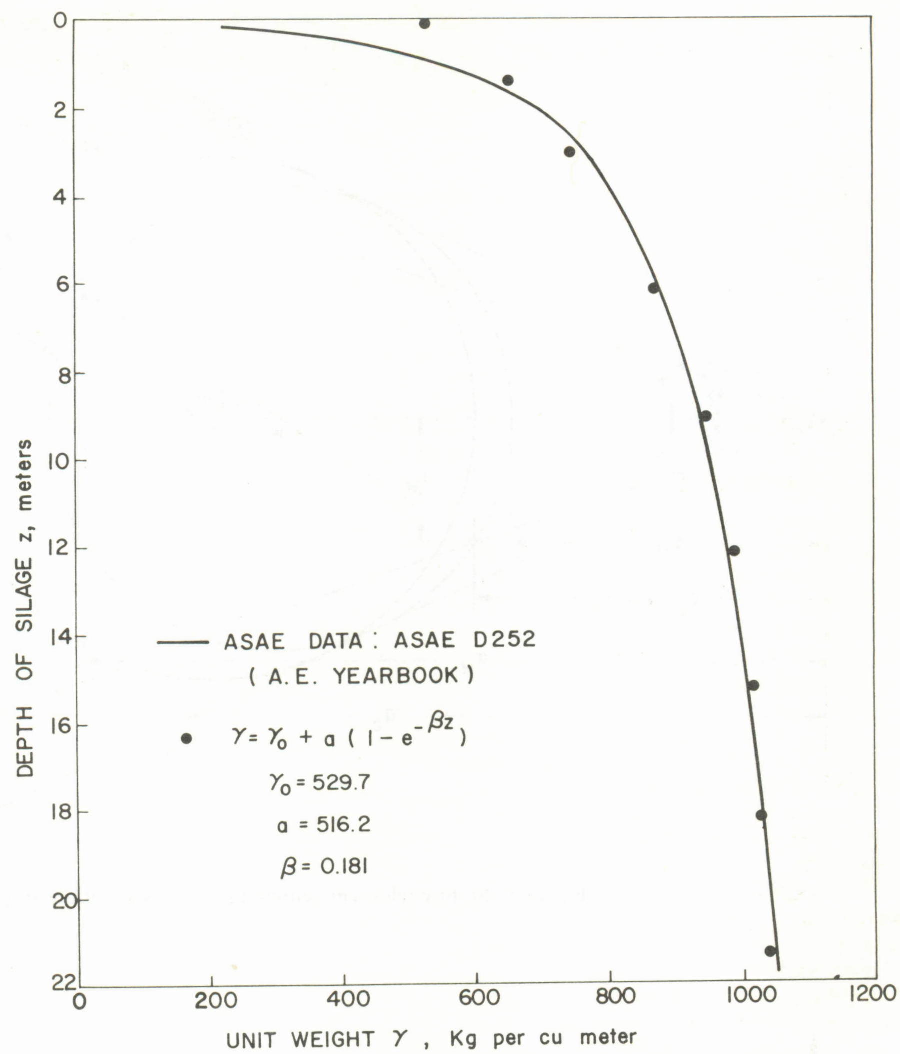

Figure 2. Unit weight of silage in tower silos.

The coefficients γ_0, a and β were determined from the experimental data with the aid of a nonlinear least-squares curve fitting computer program. The silage unit weights computed from the proposed function are plotted in Fig. 2 along with the experimental curve.

Relationships Between Pressure Components

At silo axis

The Mohr circle representing the stress conditions for an active pressure field is shown in Fig. 3. Along the axis of symmetry of a tower silo the frictional stress is zero and the major principal pressure acts in the vertical direction. For a silage material in the state of plastic equilibrium, the stress circle must touch the effective yield locus (Jenike and Johanson 1969).

From the Mohr stress circle in Fig. 3, the angle between the direction of the major principal pressure and the vertical axis is $\psi_c = 0$, and the ratio of lateral to vertical pressure is given by

$$\frac{p_c}{q_c} = \frac{1 - \sin\phi}{1 + \sin\phi}$$

where ϕ is the effective angle of internal friction to be used for cohesive solids like silage materials (Jenike 1964), and the subscript c refers to conditions at the axis of symmetry.

In a passive state of pressure, the major principal pressure acts in a horizontal direction at the silo axis and thus the major pressure angle $\psi_c = \pi/2$. The Mohr circle in Fig. 4 represents the stresses corresponding to the passive condition. The ratio of lateral to vertical pressure is now expressed by

$$\frac{p_c}{q_c} = \frac{1 + \sin\phi}{1 - \sin\phi}$$

The circumferential pressure component s can be related to the lateral pressure by utilizing the Haar-von Karman hypothesis (Haar and von Karman 1909). The hypothesis states that in axial symmetry the

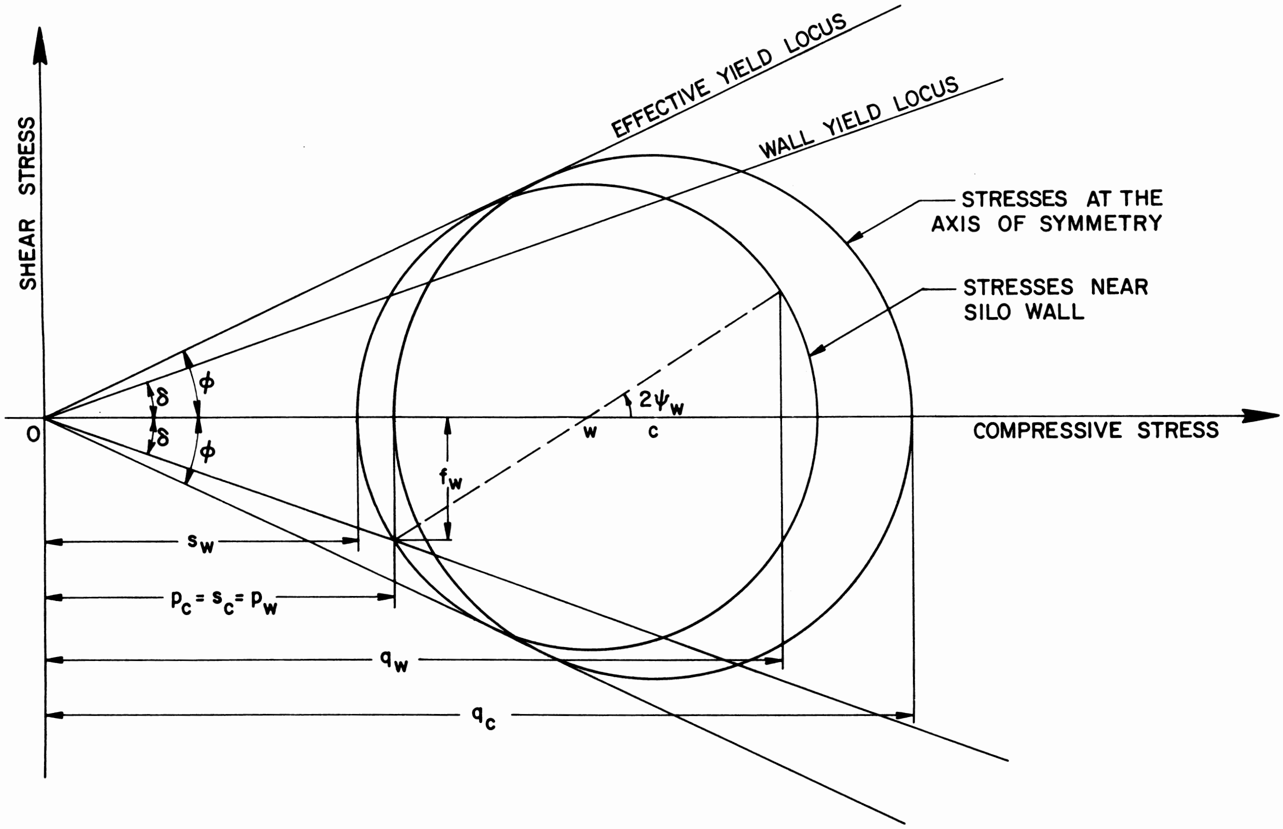

Figure 3. Mohr circles representing the active state of pressure in a tower silo.

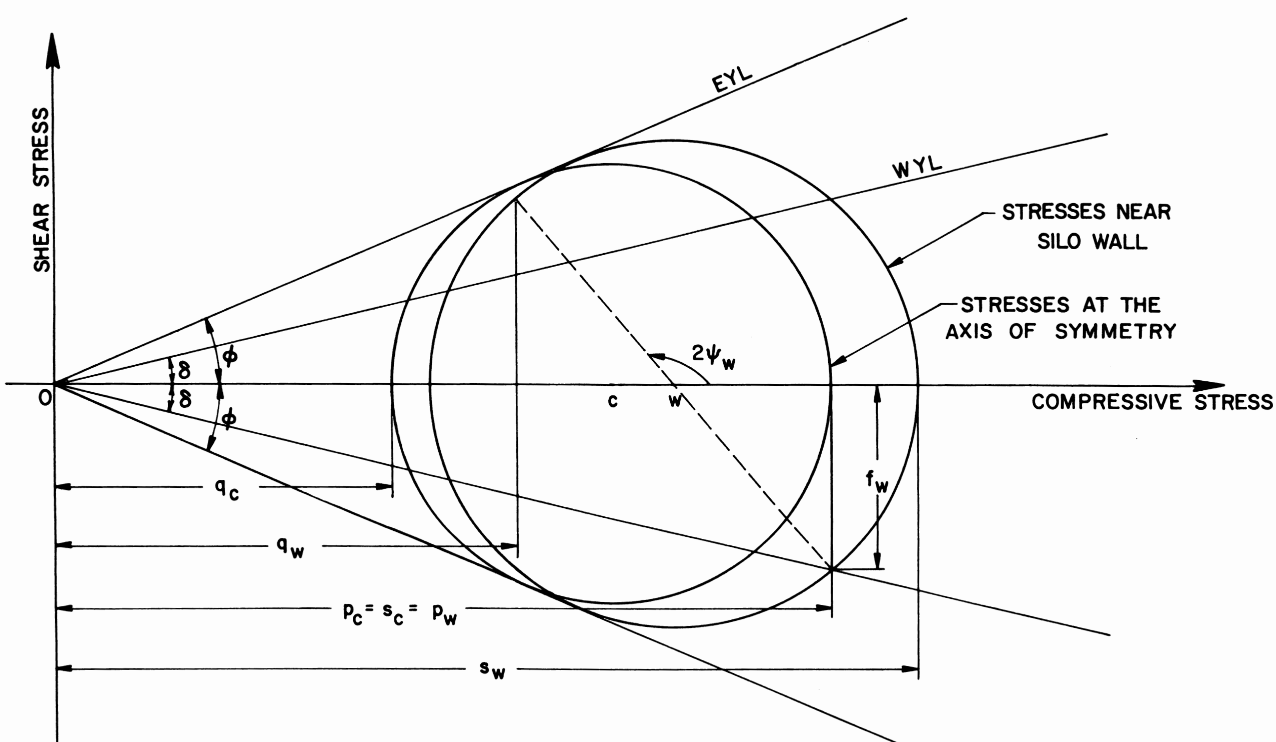

Figure 4. Mohr circles representing the passive state of pressure in a tower silo.

circumferential stress is equal to either the major or the minor stress of the meridian plane. Accordingly, s is the minor principal pressure for the active case, and the major principal pressure for the passive case.

Therefore,

$s_c = p_c = q_c\,(\frac{1 - \sin\phi}{1 + \sin\phi})$ for the active case, and

$s_c = p_c = q_c\,(\frac{1 + \sin\phi}{1 - \sin\phi})$ for the passive case.

The relationships between the lateral, vertical and circumferential pressure components for active and passive conditions along the axis of symmetry can be conveniently written in a single expression:

$$p_c = s_c = q_c\,(\frac{1 - n\sin\phi}{1 + n\sin\phi}) \quad \ldots\ldots\ldots\ldots (4)$$

in which

$n = 1$ corresponds to the active pressure field

$n = -1$ corresponds to the passive pressure field.

At silo wall

At the wall of a silo, the vertical and lateral pressures are no longer the principal pressures because wall friction introduces a vertical shear stress which distorts the pressure field. The stresses in the material adjacent to the silo wall must therefore be represented by an additional Mohr circle. From the Mohr circles representing the stresses near the wall for active and passive conditions in Figs. 3 and 4, respectively, the following relationships between the pressure components can be obtained.

$$p_w = q_w\,\frac{(1 - \sin\phi\cos 2\,\Psi_w)}{(1 + \sin\phi\cos 2\,\Psi_w)} \quad \ldots\ldots\ldots (5)$$

$$f_w = q_w\,\frac{\sin\phi\sin 2\,\Psi_w}{(1 + \sin\phi\cos 2\,\Psi_w)} \quad \ldots\ldots\ldots (6)$$

$$s_w = q_w\,\frac{(1 - n\sin\phi)}{(1 + \sin\phi\cos 2\,\Psi_w)} \quad \ldots\ldots\ldots (7)$$

where the subscript w refers to conditions at the wall.

The angle between the direction of the major principal pressure and the silo wall is given by

$$\Psi_w = (\frac{1 - n}{2})\frac{\pi}{2} + \frac{n}{2}\,[\sin^{-1}(\frac{\sin\delta}{\sin\phi}) - n\,\delta] \ldots (8)$$

where δ is the angle of friction between the silage material and the silo wall.

Method of Integral Relations

The method of integral relations, stemming from the work of Dorodnitsyn (1962), has been widely applied to solve a variety of gas dynamics problems as elucidated in Belotserkovskii and Chushkin (1965), as well as to analyze boundary layers in fluid mechanics. Savage and Yong (1970) were the first to use the integral method for the determination of stresses developed by cohesionless granular materials during slow continuous mass flow in wedge-shaped and parallel wall bins.

In this method, the integration of initial differential equations containing partial derivatives reduces to systems of ordinary differential equations. This is accomplished by subdividing the region of integration into strips and integrating the set of partial differential equations across these strips. After this, the unknown functions occurring in the integrands are replaced by interpolation expressions of the most general form.

Consider the governing pressure equilibrium equations 1 and 2 for the tower silo problem. In view of axial symmetry it is adequate to study only the half-region $0 \leqslant r \leqslant R$, where R is the radius of the silo. Multiply each of the equations 1 and 2 of the system by r and integrate with respect to the independent variable r from the axis of symmetry $r = 0$ to the silo wall $r = R$.

$$\int_0^R (r\frac{\partial q}{\partial z} + r\frac{\partial f}{\partial r} + f)\,dr = \gamma(z)\frac{R^2}{2} \quad \ldots\ldots (9)$$

$$\int_0^R (r\frac{\partial f}{\partial z} + r\frac{\partial p}{\partial r} + p - s)\,dr = 0 \quad \ldots\ldots (10)$$

In order to eliminate the partial derivatives with respect to one variable, the approximations in the variation of pressures are constructed with respect to the independent variable r. The distributions of pressure components q and s (even functions of r), and f (odd function of r) across the radius of the tower silo are assumed to be polynomials (Belotserkovskii and Chushkin 1965; Savage and Yong 1970) of the type

$$q = q_c(z) + b(z)\,(\frac{r}{R})^2 \ldots\ldots\ldots\ldots (11)$$

$$s = s_c + (\frac{s_w - s_c}{2})\,(\frac{r}{R}) \quad \ldots\ldots\ldots\ldots (12)$$

$$f = f_w(z)\,(\frac{r}{R}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Substituting equations 11, 12, 13 and the empirical relation of unit weight of silage — equation 3 — into equations 9 and 10, and integrating:

$$\frac{R}{2}\frac{dq_c}{dz} + \frac{R}{4}\frac{db}{dz} + f_w = \frac{R}{2}\,[\gamma_0 + a(1 - e^{-\beta z})] \quad \ldots\ldots\ldots\ldots\ldots (14)$$

$$\frac{R}{3}\frac{df_w}{dz} - \frac{3}{4}s_c - \frac{1}{4}s_w + p_w = 0 \quad \ldots\ldots\ldots (15)$$

Solution of Equations

At the silo wall $r = R$, and the vertical pressure at the silo axis from equation 11 is consequently

$$q_c = q_w - b(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Insert equations 6 and 16 into equation 14 to obtain

$$\frac{R}{2}\frac{dq_w}{dz} + \frac{\sin\phi\sin 2\,\Psi_w}{(1 + \sin\phi\cos 2\,\Psi_w)}q_w - \frac{R}{4}\frac{db}{dz} = \frac{R}{2}\,[\gamma_0 + a(1 - e^{-\beta z})] \quad \ldots\ldots\ldots\ldots (17)$$

By substituting equations 4, 5, 6, 7, 16 in equation 15 and with some rearrangement, the following equation is obtained

$$\frac{4R\sin\phi\sin 2\,\Psi_w\,(1 + n\sin\phi)}{9(1 + \sin\phi\cos 2\,\Psi_w)(1 - n\sin\phi)}\frac{dq_w}{dz} + \frac{(n - \cos 2\,\Psi_w)(n\sin^2\phi + 7\sin\phi)}{3(1 - n\sin\phi)(1 + \sin\phi\cos 2\,\Psi_w)}q_w + b = 0 \quad \ldots\ldots\ldots\ldots (18)$$

Differentiation of equation 18 with respect to z yields

$$\frac{4R\sin\phi\sin 2\,\Psi_w\,(1 + n\sin\phi)}{9(1 + \sin\phi\cos 2\,\Psi_w)(1 - n\sin\phi)}\frac{d^2q_w}{dz^2} + \frac{(n - \cos 2\,\Psi_w)(n\sin^2\phi + 7\sin\phi)}{3(1 - n\sin\phi)(1 + \sin\phi\cos 2\,\Psi_w)}\frac{dq_w}{dz} + \frac{db}{dz} = 0 \quad \ldots\ldots\ldots\ldots (19)$$

Elimination of db/dz between equations 17 and 19 leads to a nonhomogeneous linear second-order differential equation with constant coefficients.

$$U\frac{d^2q_w}{dz^2} + V\frac{dq_w}{dz} + W\,q_w = \gamma_0 + a(1 - e^{-\beta z}) \quad \ldots\ldots\ldots\ldots (20)$$

where

$$U = \frac{2R\,(1 + n\sin\phi)\sin\phi\sin 2\,\Psi_w}{9(1 - n\sin\phi)\,(1 + \sin\phi\cos 2\,\Psi_w)}$$

$$V = 1 + \frac{(n - \cos 2\,\Psi_w)\,(n\sin^2\phi + 7\sin\phi)}{6(1 - n\sin\phi)\,(1 + \sin\phi\cos 2\,\Psi_w)}$$

$$W = \frac{2\sin\phi\sin 2\,\Psi_w}{R\,(1 + \sin\phi\cos 2\,\Psi_w)} \quad \ldots\ldots\ldots\ldots (21)$$

By applying the method of undetermined coefficients to equation 20 and using the principle of superposition, a closed-form solution is obtained.

$$q_w = C_1e^{k_1z} + C_2e^{k_2z} - \frac{ae^{-\beta z}}{(U\beta^2 - V\beta + W)} + \frac{(\gamma_0 + a)}{W} \quad \ldots\ldots\ldots (22)$$

In the cases of practical interest the roots k_1 and k_2 are real and distinct. The arbitrary constants C_1 and C_2 are determined from the following boundary conditions.

For a horizontal pressure-free top surface:

(i) At $z = 0$; $q_w = 0$

(ii) From equations 16 and 18

At $z = 0$; $b = 0$; $\frac{dq_w}{dz} = 0$

The vertical pressure at the silo wall is now given by

$$q_w = C_1e^{k_1z} + C_2e^{k_2z} + C_3e^{-\beta z} + C_4 \ldots (23)$$

where

$$k_{1,2} = -\frac{V}{2U} \pm \frac{V}{2U}\sqrt{(1 - \frac{4UW}{V^2})}$$

$$C_1 = \frac{C_3\,(k_2 + \beta) + C_4k_2}{(k_1 - k_2)}$$

$$C_2 = \frac{C_3\,(k_1 + \beta) + C_4k_1}{(k_2 - k_1)}$$

$$C_3 = -\frac{a}{(U\beta^2 - V\beta + W)}$$

$$C_4 = \frac{(\gamma_0 + a)}{W}$$

The vertical pressure at the axis of symmetry of the tower silo is obtained by introducing equation 18 into equation 16.

$$q_c = \frac{4R(1 + n \sin \phi) \sin \phi \sin 2\Psi_w}{9(1 - n \sin \phi)(1 + \sin \phi \cos 2\Psi_w)} \frac{dq_w}{dz} + \left[1 + \frac{(n - \cos 2\Psi_w)(n \sin^2 \phi + 7 \sin \phi)}{3(1 - n \sin \phi)(1 + \sin \phi \cos 2\Psi_w)}\right] q_w \quad \ldots\ldots\ldots\ldots\ldots (24)$$

Rearrangement of equation 24 yields

$$q_c = 2U \frac{dq_w}{dz} + (2V - 1) q_w \quad \ldots\ldots\ldots\ldots (25)$$

The rate of change of vertical wall pressure with depth is given by

$$\frac{dq_w}{dz} = C_1 k_1 e^{k_1 z} + C_2 k_2 e^{k_2 z} + \frac{a\beta e^{-\beta z}}{(U\beta^2 - V\beta + W)} \quad \ldots\ldots\ldots\ldots\ldots (26)$$

The complete pressure field within a tower silo can thus be established by the aid of equations 4, 5, 6, 7, 23, 25, and 26.

The tedious numerical procedure required to obtain the complete pressure field within a tower silo suggests the use of a digital computer. Therefore, a computer program was developed to calculate all the silage pressure components for the active and passive pressure fields from the equations derived in the preceding sections. The complete computer program and numerical solutions for specific sets of system parameters are available in Negi (1975).

DESIGN CONSIDERATIONS

From the silo designer's standpoint, the practical question at the present time is: what pressures should the silo walls be designed for? It has been elucidated that two types of pressure fields can develop in a silo. During filling and subsequent state of rest, an active pressure field develops, provided that the silage is charged into the silo without significant impact (Jenike et al. 1973). The pressure condition in a silo during withdrawal of silage is highly dependent on the location and the design of the unloading device. When discharging is carried out by a silo unloader located on top of the stored material, there is essentially no motion of the silage mass in the silo. It is believed that with such a top unloader system, an active state of pressure prevails throughout the storage period. Accordingly, the order of the pressures which will be developed in silage silos equipped with top unloaders can be computed using the proposed theory for the active case.

The bottom unloading of a silo generates mass or funnel flow of silage, depending on the design of the unloader system. During mass flow, the motion of the contained material is sufficiently slow and close to steady state for the inertial forces to be negligible (Jenike and Johanson 1968). Consequently, the stress state is similar to the active case except that the kinematic angle of wall friction should be used to account for the motion of the silage mass. The use of this parameter will result in somewhat higher wall pressures than those encountered in the static state.

In the event of funnel flow, the overpressures at the plane of transition from peaked to arched conditions are subdued by the nonflowing mass (Jenike and Johanson 1969), and thus the resulting wall pressures are attenuated. In this instance, a stress condition somewhat in between the active case and the passive case may develop. However, it should be noted that silage is cohesive and not sufficiently free-flowing material and is thus capable of forming a self-supporting obstruction to flow. It can be conceived that when a stable arch of silage collapses it will cause prominent dynamic effects and impose momentary overpressures on the silo. Therefore, under these conditions, silage pressures can be calculated using the present theory for the passive case. In passing, it is noted that the active and passive cases are synonymous with the static and dynamic conditions, respectively.

In addition to silage pressures, the following loads caused by the multiplicity of factors must be taken into account in the detail design of silos:

1. Extraneous loads imposed by the filling, spreading and unloading equipment.
2. Reduction of the wall friction effect and the attendant increase in lateral pressures as a consequence of vibrations from the equipment mentioned above and/or from vehicular usage in the vicinity of the silo.
3. Axial compression and bending moments induced by the wind pressure in the leeward walls, which are of importance when the silo is empty.
4. Effects of temperature variations, particularly in geographic locations where there are marked temperature fluctuations. For instance, in the province of Quebec, the temperature varies typically between -30 C and 30 C. Thus, thermal stresses would be of critical significance.
5. Loads applied during and after tensioning of steel hoops around the outside of a concrete stave silo.
6. Secondary stresses due to shrinkage of the concrete and creep due to sustained loading. The amount of shrinkage depends on the age of the concrete and on the method of curing. Creep is affected by ambient conditions, properties of concrete and its matrix, loading history and especially duration of sustained load. The formation and propagation of cracks in reinforced concrete silo walls are believed to be caused by these so-called "secondary" factors.
7. Effects of the discontinuity at the top of the wall with the roof of the silo.
8. Effects of slight imperfections in the shape or finish of cylindrical silos, such as ledges at the level of girth seams in a welded steel silo, or inevitable deviations from a perfect cylinder in a slip-form concrete silo. It has been reported that boundary layers developed during flow at these distortions cause pressure oscillations and impose sharp overpressure on the walls (Jenike et al. 1973).

Footings for tower silos must be designed to minimize differential settlement of supports, and to prevent excessive settlement and tilting of the silo. The self-weight of the structure, storage load and aforementioned external forces must be considered in determining the size of, and the amount of reinforcement in the footing. In addition, the maximum pressure must not exceed the safe bearing capacity of the soil. Comments regarding problems associated with foundations for tower silos as well as allowable bearing capacity of clay soils are to be found in a recent publication by Bozozuk (1974).

Finally, the design engineer must keep in mind that, while the silo is an integral part of the farming system, the choice of size may be governed as much by the vertical load on the silo wall as by the required depth to be removed per day.

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