A new approach to solving a transient-state drainage equation

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Uziak, J. and Chieng, S. 1988. **A new approach to solving a transient-state drainage equation.** Can. Agric. Eng. 30: 319–321. A new solution to the transient-state drainage equation for drainage computations was presented. This paper reports the development of this solution by using Galerkin's Method and a comparison of it with the well-known Glover's and Tapp and Moody's solutions. It was concluded that the proposed equation gives better results than Glover's or Tapp and Moody's equations for normalized time less than 0.01.

TRANSIENT-STATE DRAINAGE EQUATION

Within the past 30 yr, a considerable amount of research on drainage problems has been done. So far, drainage problems have been divided into steady-state and transient-state flow conditions. A steady state exists when the boundaries and flow rates of a system do not change with time. Otherwise, a transient state exists. As steady state seldom exists under the actual field conditions, solutions by using the transient state condition should be adopted. In transient flow, the position of the water table between two parallel drains (see Fig. 1) can be given by the following differential equation (Dumm 1960, 1964):

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial X^2} \tag{1}$$

where:

- y = water table height above the datum;
- t = time (d);
- α = $K.D/f$;
- K = soil hydraulic conductivity (m/d);
- D = $d + y_0/2$;
- D = average depth of flow region (m);
- d = depth to impermeable layer below drain (m);
- y_0 = water table height above drains at mid-spacing as shown in Fig. 1 at time t = 0, (m);
- f = drainable porosity, percentage by volume;
- X = outward horizontal distance from drain (m);
- S = drain spacing (m);

The boundary conditions for equation (1) are:

$$y(0,t) = y(S,t) = 0 \tag{2}$$

An initial condition (or initial water table profile) of a fourth-degree parabola was suggested by Tapp and Moody (Dumm 1960, 1964) and has been evaluated by the United States Bureau of Reclamation (Luthin 1978) as follows:

$$y(X,0) = \frac{8y_0}{S^4}(S^3X - 3S^2X^2 + 4SX^3 - 2X^4) \tag{3}$$

Although this parabola only approximates the initial water table

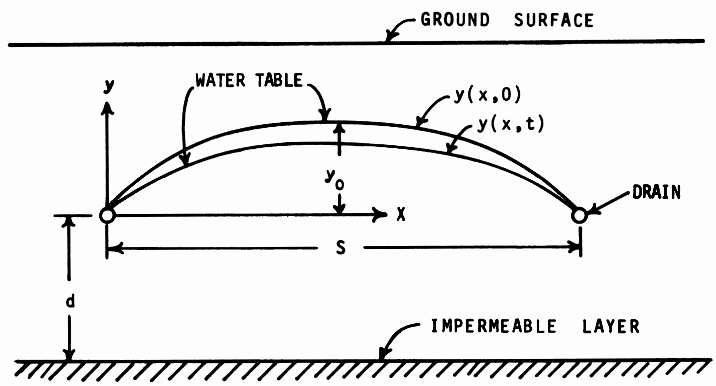

Figure 1. Geometry of drainage problem.

shape, the U.S. Bureau of Reclamation has used it in modifying the well-known Glover's equation as a solution to Eq. 1 under Eq. 2 boundary condition (Van Schilfgaarde 1974) and according to Dumm (1960, 1964) can take the following form:

$$y(X,t) = \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{n^2\pi^2-8}{n^5}\right)\exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right)\sin\left(\frac{n\pi X}{S}\right) \tag{4}$$

Substituting $X = S/2$ into Eq. 4, the following (Glover's) working equation is obtained for the height of the water table at mid-spacing:

$$y\left(\frac{S}{2},t\right) = \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\left(\frac{n^2\pi^2-8}{n^5}\right)\exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right) \tag{5}$$

Tapp and Moody, as reported by Dumm (1960), suggested that an approximate solution can be obtained by taking only the first term of the series in Eq. 5:

$$y\left(\frac{S}{2},t\right) \approx \frac{192y_0}{\pi^5}(\pi^2-8)\exp\left(-\frac{\pi^2\alpha t}{S^2}\right) \tag{6}$$

ALTERNATIVE SOLUTION PROPOSED

The Galerkin's method was applied to solve the differential Eq. 1 with boundary and initial conditions given in Eqs. 2 and 3, respectively.

Galerkin's method approximates the solution of a differential equation

$$L(\mu) = 0 \tag{7}$$

with the series

$$\tilde{\mu} = \sum_i a_i \theta_i \tag{8}$$

The functions θ_i must be linearly independent and differentiable to the extent that all terms in the differential equation and boundary conditions can be obtained. For a one-dimensional system, the set of functions could be either:

$$\theta_0 = 1 \text{ and } \theta_i = \sin(i\,\pi\,X) \qquad i = 1,2,3,\ldots \tag{9}$$

(Carson et al. 1979) or:

$$\theta_0 = 1 \text{ and } \theta_i = x^i \qquad i = 1,2,3,\ldots \tag{10}$$

(Mikhlin and Smolitskiy 1967). When the approximate-solution is substituted into the differential Eq. 7, the equation becomes:

$$L(\tilde{\mu}) = \epsilon \tag{11}$$

where ϵ is the error in the approximation. The parameters a_i in Eq. 8 can be determined by using orthogonal conditions for the error of approximation (ϵ) and trial functions ($0_j, j = 1,2,3,\ldots$) as suggested by Mikhlin and Smolitskiy (1967):

$$\int_\Omega \epsilon\theta_j d\Omega = 0 \qquad j = 1,2,3,\ldots \tag{12}$$

If the trial function satisfies the boundary conditions but not the differential equation, Ω is the region interior to the boundary. It defines the boundary if it satisfies the differential equation but not the boundary conditions.

In using the Galerkin's method, a trial function satisfying the two conditions (Eq. 2 and 3) is chosen as:

$$y(X,t) = \frac{8y_0}{S^4}(S^3X - 3S^2X^2 + 4SX^3 - 2X^4) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n X}{S}\right) \tag{13}$$

and the error term can be obtained as:

$$\epsilon = \sum_{n=1}^{\infty}\left(\frac{da_n}{dt} \sin\left(\frac{\pi n X}{S}\right)\right) - \frac{48\alpha y_0}{S^4}(-4X^2 + 4SX - S^2) + \frac{\pi^2\alpha}{S^2}\sum_{n=1}^{\infty} n^2 a_n \sin\left(\frac{\pi n X}{S}\right) \tag{14}$$

The parameters a_n ($n = 1,2,3,\ldots$) in the above equation can be obtained from the orthogonal conditions (i.e. Eq. 12) which constitute a set of the following equations:

$$\int_0^S \epsilon \sin\frac{\pi m}{S} X dX = 0 \qquad \text{where } m = 1,2,3,\ldots \tag{15}$$

This gives Eq. 12 the form of:

$$\int_0^S \Big[\sum_{n=11}^{\infty}\left(\frac{da_n}{dt} \sin\frac{\pi n X}{S}\right) - \frac{48\alpha y_0}{S^4}(-4X^2 + 4SX - S^2) + \frac{\pi^2\alpha}{S^2}\sum_{n=1}^{\infty} n^2 a_n \sin\left(\frac{\pi n X}{S}\right)\Big] \sin\frac{\pi m X}{S} dX = 0$$

where $m = 1,2,3,\ldots$ (16)

The result of the integration of Eq. 16 is the set of linear differential equations:

$$\frac{da_m}{dt} + \frac{\pi^2 m^2 \alpha}{S^2} a_m - \frac{192 y_0 \alpha}{S^2 \pi m}\left(\frac{8}{\pi^2 m^2} - 1\right) = 0$$

where $m = 1,3,5,\ldots$ (17)

These equations are independent and could be solved separately to give the following results:

$$a_m = 192y_0\left(\frac{8 - \pi^2 m^2}{m^5\pi^5}\right)\left[1 - \exp\left(-\frac{m^2\pi^2\alpha t}{S^2}\right)\right]$$

where $m = 1,3,5,\ldots$ (18)

Substituting Eq. 18 into Eq. 13, gives the solution for Eq. 1 as follows:

$$y(X,t) = \frac{8y_0}{S^4}(S^3X - 3S^2X^2 + 4SX^3 - 2X^4) - \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\frac{n^2\pi^2 - 8}{n^5}\left[1 - \exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right)\right]\cdot \sin\frac{n\pi X}{S} \tag{19}$$

DISCUSSION OF THE SOLUTIONS

As mentioned before, the solution of the transient-state differential equation is often expressed in the form of Eq. 4 as reported by Dumm (1960, 1964). It should be noted that Eq. 4 does not satisfy the initial condition (i.e., Eq. 3), while Eq. 19 proposed in this paper does satisfy the initial condition "precisely." Equation 4 could obtain the precision of Eq. 19 by expanding the initial condition into the Fourier series as follows:

$$y(X,O) = \frac{8y_0}{S^4}(S^3X - 3S^2X^2 + 4SX^3 - 2X^4) \approx \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\pi^2 n^2 - 8}{n^5}\sin\frac{n\pi X}{S} \tag{20}$$

When Eq. 20 is substituted into Eq. 19, the following solution is obtained:

$$y(X,t) = \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{\pi^2 n^2 - 8}{n^5}\right)\exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right)\sin\frac{n\pi X}{S} \tag{21}$$

We can see that Eq. 21 is exactly the same as Eq. 4. This indicates that solution in the form of Eqs. 4 and 21 could be treated as the approximation of Eq. 19 proposed in this paper. For the height of the water table at mid-spacing ($X = S/2$), Eq. 19 becomes:

$$y\left(\frac{S}{2}, t\right) = y_0 - \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\left(\frac{n^2\pi^2 - 8}{n^5}\right)\left[1 - \exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right)\right] \tag{22}$$

since

$$\frac{192}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\left(\frac{n^2\pi^2 - 8}{n^5}\right) = 1$$

Therefore, Eq. 22 can be simplified to:

$$y\left(\frac{S}{2}, t\right) = \frac{192y_0}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\left(\frac{n^2\pi^2 - 8}{n^5}\right)\exp\left(-\frac{n^2\pi^2\alpha t}{S^2}\right) \tag{23}$$

Table I. Relationship between y/y_0 and $\alpha t/S^2$ at mid-spacing by using different equations

$\alpha t/S^2$†	Eq. 5‡	Eq. 22§	Eq. 6¶	Eq. 24‖
	y/y_0			
0.0001	1.0000	1.0000	1.1718	0.9988
0.0002	1.0000	1.0000	1.1707	0.9977
0.0003	1.0000	1.0000	1.1635	0.9965
0.0004	1.0000	1.0000	1.1684	0.9954
0.0005	0.9999	0.9999	1.1672	0.9942
0.0006	0.9999	0.9999	1.1661	0.9931
0.0007	0.9999	0.9999	1.1643	0.9919
0.0008	0.9999	0.9999	1.1638	0.9908
0.0009	0.9998	0.9998	1.1626	0.9836
0.0010	0.9998	0.9998	1.1615	0.9885
0.002	0.9992	0.9992	1.1501	0.9771
0.003	0.9983	0.9983	1.1388	0.9658
0.004	0.9969	0.9969	1.1276	0.9546
0.005	0.9952	0.9952	1.1165	0.9435
0.006	0.9931	0.9931	1.1056	0.9326
0.007	0.9906	0.9906	1.0347	0.9217
0.008	0.9877	0.9877	1.0840	0.9109
0.009	0.9845	0.9845	1.0733	0.9003
0.010	0.9808	0.9808	1.0628	0.8898
0.02	0.9279	0.9279	0.9629	0.7899
0.03	0.8579	0.8579	0.8724	0.6994
0.04	0.7844	0.7844	0.7904	0.6174
0.05	0.7137	0.7137	0.7161	0.5431
0.06	0.6478	0.6478	0.6488	0.4758
0.07	0.5874	0.5874	0.5878	0.4148
0.08	0.5324	0.5324	0.5326	0.3596
0.09	0.4825	0.4825	0.4825	0.3095
0.10	0.4372	0.4372	0.4372	0.2642
0.2	0.1629	0.1630	0.1629	−0.0101
0.3	0.0607	0.0607	0.0607	−0.1123
0.4	0.0226	0.0226	0.0226	−0.1504
0.5	0.0084	0.0084	0.0084	−0.1646
0.6	0.0031	0.0032	0.0031	−0.1633
0.7	0.0012	0.0012	0.0012	−0.1718
0.8	0.0004	0.0004	0.0004	−0.1726
0.9	0.0002	0.0002	0.0002	−0.1728
1.0	0.0001	0.0001	0.0001	−0.1729

†Normalized time.
‡Glover's solution by using Fourier Series reported by Dumm (1960, 1964).
§Glover's solution by using Galerkin's Method proposed in this paper.
¶Tapp and Moody's equation.
‖Approximation of the solution of Galerkin's Method by taking first term only.

It can be seen that Eq. 23 is exactly the same as Eq. 5 (Glover's) mentioned above. If only the first term of the approximation in Eq. 22 is taken, the equation becomes:

$$y\left(\frac{S}{2}, t\right) \approx y_0 - \frac{192 y_0}{\pi^5}(\pi^2 - 8)\left[1 - \exp\left(-\frac{\pi^2 \alpha t}{S^2}\right)\right] \quad (24)$$

Equation 24 is an alternate solution to Eq. 6 or Tapp and Moody's approximation. This shows that Galerkin's method can be used to obtain another solution for the transient-flow drain-spacing equation.

In subsurface drainage design, spacings are calculated by choosing a water table height above the drains at mid-spacing. In order to compare the proposed drain-spacing equation (i.e., Eq. 24) to Glover's and Tapp and Moody's equations, ratios of $y(S/2,t)/y_0$ were calculated for different normalized time, $\alpha t/S^2$ (Dumm 1964), as given in Table I. It can be seen from Table I that while both the proposed equation (Eq. 24) and the Tapp and Moody's equation give the approximate solution to the exact solution, the proposed equation gives better results when $\alpha t/S^2$ value is smaller than 0.01 and the Tapp and Moody's equation gives the better approximation when $\alpha t/S^2$ is greater than 0.01. This means that the proposed equation should be used to obtain the solution when the normalized time is small as the initial condition is an important factor to be considered in the approximation.

CONCLUSION

The transient-flow differential equation was solved by using Galerkin's method. This solution satisfied the initial condition "precisely" and it had the same precision as the solution with the Fourier series expansion of the initial condition. Owing to this precision, the proposed equation gave a better solution to the differential equation than the solution (i.e., Eq. 4) reported by Dumm (1960, 1964).

It should be noted that none of the equations (Eq. 24) proposed in this paper and Tapp and Moody's equation) gave the "exact" solution to differential Eq. 1. As the proposed solution satisfied the initial conditions, it gave better results than Tapp and Moody's equation when the normalized time ($\alpha t/S^2$) was smaller than 0.01 as shown in Table 1. It is recommended that the proposed equations (Eq. 24) and Tapp and Moody's equation should be used to calculate the drain spacing for transient flow drainage at different normalized time steps. The former should be used when $\alpha t/S^2$ is smaller than 0.01 and the latter will give better approximation when $\alpha t/S^2$ is greater than 0.01.

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