Drain-spacing formula for transient-state flow with ellipse as an initial condition

JACEK UZIAK[1] and SIETAN CHIENG[2]

[1]*Department of Machine Theory and Automatics, Academy of Agriculture, Lublin, Poland; and* [2]*Bio-Resource Engineering Department, University of British Columbia, Vancouver, BC, Canada V6T 1W5.* *Received 1 June 1988, accepted 12 December 1988.*

Uziak, J. and Chieng, S. 1989. **Drain-spacing formula for transient state flow with ellipse as an initial condition.** Can. Agric. Eng. **31**: 101–105. A new solution of the differential equation describing the drainage problem was presented. An initial condition in the form of an ellipse, approximated by a negative exponential function, was used. From this solution, a new formula was proposed for drain-spacing calculations. It was found that the proposed formula is more general for drain-spacing calculations. Two well-known drain-spacing formulae, Glover's equation (Van Schilfgaarde 1974), and Glover-Dumm's (Dumm 1960, 1964) equation, are two special cases of the proposed formula.

INTRODUCTION

Within the past three decades, a considerable amount of research on drainage problems has been done. So far, drainage problems have been divided into steady-state and transient-state flow conditions. A steady state exists when the boundaries and flow rates of a system do not change with time. Otherwise, a transient state exists. As steady state seldom exists under actual field conditions, solutions using the transient-state condition should be adopted. However, owing to their simplicity, steady-state equations based on elliptical initial conditions have been widely used in subsurface drainage design. Currently, there is no transient-state drain-spacing equation based on elliptical initial conditions. Drain-spacings calculated from steady and nonsteady state equations cannot be meaningfully compared. This study addresses this need by using the ellipse-initial condition in transient-state flow and developing a new drain-spacing equation.

TRANSIENT FLOW

Transient or nonsteady-state conditions occur when the ground water table fluctuates with time; therefore, the hydraulic head is changing constantly. The linearized differential equation for transient flow, as derived on the basis of Dupuit-Forchheimer assumptions, can be expressed as (Luthin 1978):

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where:

α = $K.D/f$
K = soil-saturated hydraulic conductivity,
D = average depth of flow region = $d + Y_0/2$,
d = depth to impermeable layer below drain,
Y_0 = water table height at mid-spacing as shown in Fig. 1,
f = drainable porosity or specific yield,
t = time, and
Y = water table height above the datum.

The solution of Eq. 1 depends on the initial and boundary conditions. The boundary conditions are relatively simple and are generally set up as follows:

$$Y = 0 \quad (\text{for } x = 0 \text{ and } X = L) \tag{2}$$

where L = drain spacing.

The initial condition is much more complicated and three types of initial water table shapes, as shown in Fig. 2 (Dass and Morel-Seytoux 1974), are usually assumed:

(a) A constant water table height, Y_0, exists everywhere between the two adjacent drains except above the drains where the water table abruptly drops to zero (to drain level).

$$Y = Y_0 \qquad (\text{for } 0 < X < L \text{ where } t = 0) \tag{3}$$

(b) The water table shape is a fourth degree parabola with the following form:

$$Y = \frac{8Y_0}{L^4}(L^3X - 3L^2X^2 + 4LX^3 - 2X^4) \; (\text{for } t = 0) \tag{4}$$

(c) The water table is a parabola with the inflexion point given by the expression:

$$Y = \frac{16Y_0}{L^4} X^2 (L-X)^2 \qquad (\text{for } t = 0) \tag{5}$$

The initial water table shape with a fourth degree parabola expressed by Eq. 4 is most widely used. Using this initial condition and the boundary condition in the form of Eq. 2, the solution of Eq. 1 in the following form is obtained (Luthin 1978):

$$Y = \frac{192Y_0}{\pi^5} \sum_{n=1,3,5,\ldots}^{\infty} \frac{n^2\pi^2 - 8}{n^5} \exp\left(-\frac{n^2\pi^2\alpha t}{L^2}\right) \sin\frac{n\pi X}{L} \tag{6}$$

At mid-spacing, $X=L/2$, the water table height can be expressed as:

$$Y_m = \frac{192}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{n^2 - 8/\pi^2}{n^5} \exp\left(-\frac{\pi^2 n^2 \alpha t}{2}\right) \tag{7}$$

ELLIPSE AS AN INITIAL CONDITION

In the steady-state condition it is assumed that the recharge and drainage rates are equal, and the hydraulic head does not vary with time. This situation is described by Hooghoudt (Luthin 1978) as an ellipse equation with a semi-major axis of $\frac{K}{R}\left(d^2 + \frac{8L^2}{4K}\right)$ and a semi-minor axis of $\left(d^2 + \frac{RL^2}{4K}\right)$

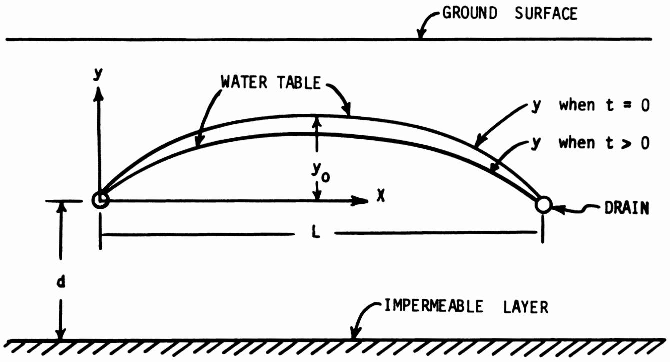

Figure 1. Definition sketch for drainage problem.

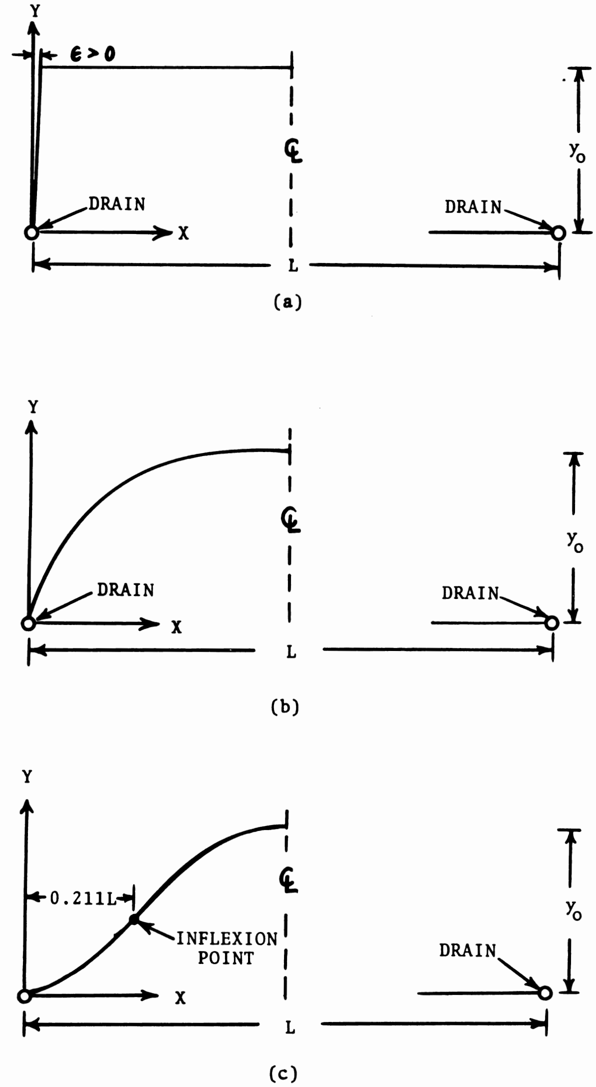

Figure 2. Three initial water table conditions studied by Dass and Morel-Seytous.

where R is the drainage coefficient or recharge rate. These are shown in Fig. 1. The steady-state condition becomes a nonsteady-state condition when the recharge and drainage rates are not balanced. Hooghoudt's equation and other steady-state equations have been used widely in subsurface drainage design. It therefore seems logical to assume the ellipse as an initial condition for transient-state conditions. This enables the comparison of drain-spacings calculated from two different approaches (i.e., steady and nonsteady) based on the same initial condition (i.e., ellipse). The equation of this ellipse-condition can be expressed as:

$$\frac{(X-L/2)^2}{\dfrac{L^2(Y_0+d)^2}{4[(Y_0+d)^2-d^2]}} + \frac{(Y+d)^2}{(Y_0+d)^2} = 1 \tag{8}$$

Unfortunately, because of the mathematical complications, it is extremely difficult to obtain the analytical solution of Eq. 1 in the normal manner using an ellipse as the initial condition. For this reason the authors approximate the ellipse by the well-established negative-exponential function as:

$$Y = A(1 - \exp(-BX)) \qquad (\text{for } 0 \le X \le L/2) \tag{9a}$$

$$Y = A(1 - \exp(-B(L-X)) \qquad (\text{for } L/2 \le X \le L) \tag{9b}$$

From Eqs. 8, 9a and 9b the following expressions are obtained:

$$A = \frac{Y_0}{1 - \exp\left[-\dfrac{(Y_0+d)^2}{d^2}\right]} \tag{10}$$

$$B = \frac{2(Y_o+d)^2}{d^2 \cdot L} \tag{11}$$

Figure 3 compares the shapes of ellipses (Eq. 8) and negative-exponential functions (Eqs. 10 and 11). The agreement between the curves depends on the value of d/Y_0. Comparisons were done for $d/Y_0 = 0$ to $d/Y_0 = \infty$. The best agreement is found for $d/Y_0 = 2$. Although the ratio of d/Y_0 could vary from 0 to infinity in theory, a range of 0–5 is more frequently encountered in practice. As $d/Y_0 = 2$ is about half-way between 0 and 5, it is felt that the proposed negative-functions can be used to adequately approximate the ellipse.

Substituting equations 10 and 11 into Eqs. 9a and 9b, yields:

$$Y = \frac{Y_0}{1 - \exp\left[-\left(\dfrac{Y_0+d}{d}\right)^2\right]} \cdot \left[1 - \exp\left(-\frac{2(Y_0+d)^2 X}{Ld^2}\right)\right] \quad (\text{for } 0 \le X \le L/2) \tag{12a}$$

$$Y = \frac{Y_0}{1 - \exp\left[-\left(\dfrac{Y_0+d}{d}\right)^2\right]} \cdot \left[1 - \exp\left(-\frac{2(Y_0+d)^2(L-X)}{Ld^2}\right)\right] \quad (\text{for } L/2 \le X \le L) \tag{12b}$$

SOLUTION OF TRANSIENT STATE EQUATION

A solution of the transient state equation (Eq. 1) which satisfies the boundary conditions (Eq. 2) and proposed ellipse initial condition (Eqs. 12a and 12b), is obtained as follows:

$$Y = \frac{4Y_0}{1-\exp\left[-\left(1+\frac{Y_0}{d}\right)^2\right]} \sum_{n=1,3,5,\ldots}^{\infty} \left\{\frac{1}{n\pi} - \frac{1}{4\left(1+\frac{Y_0}{d}\right)^4 + n^2\pi^2}\left[n\pi - (-1)^{\frac{n-1}{2}} \cdot 2\left(1+\frac{Y_0}{d}\right)^2 \cdot \exp\left[-\left(1+\frac{Y_0}{d}\right)^2\right]\right]\right\} \cdot \exp\left(-\frac{\alpha\pi^2 n^2 t}{L^2}\right) \cdot \sin\left(\frac{n\pi X}{L}\right) \tag{13}$$

Equations 12a and 12b which have been used as initial conditions for Eq. 13 are an approximation of an ellipse given by Eq. 8. It can be seen in Fig. 3 that the agreement between the original ellipse's function and the proposed approximation appears to be very good. Therefore, Eq. 13 can be treated as a solution of Eq. 1 with an ellipse as the initial water table shape. Since the main interest is in the height of the water table at the midpoint between the drains, we can obtain the following expression for y, at $x = L/2$:

$$Y_m = \frac{4Y_0}{1-\exp\left[\left[\left(1+\frac{Y_0}{d}\right)^2\right]\right]} \cdot \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \left\{\frac{1}{n\pi} - \frac{1}{4\left(1+\frac{Y_0}{d}\right)^4 + n^2\pi^2}\left[n\pi - (-1)^{\frac{n-1}{2}} \cdot 2\left(1+\frac{Y_0}{d}\right)^2 \exp\left(-\left(1+\frac{Y_0}{d}\right)^2\right)\right]\right\} \cdot \exp\left(-\frac{\alpha\pi^2 n^2 t}{L^2}\right) \tag{14}$$

where Y_m is the water table height above drains at mid-spacing, at time t.

Figure 4 shows the relationship between the dimensionless parameters Y/Y_0 and KDt/fL^2 for different ratios of d/Y_0. As a comparison, the same relationship for the U.S. Bureau of Reclamation's equation (Eq. 6) is also included. It can be seen from Fig. 4 that when d/Y_0 increases the time required for water table height to drop from Y_0 to Y decreases. When the drains are placed on the impermeable layer (d=0), the value of d/Y_0 is zero and Eq. 13 becomes:

$$Y = \frac{4Y_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \exp\left(-\frac{\alpha\pi^2 n^2 t}{L^2}\right) \cdot \sin\left(\frac{n\pi X}{L}\right) \tag{15}$$

and the water table height at mid-spacing can be obtained as:

$$Y_m = \frac{4Y_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n} \exp\left(-\frac{\alpha\pi^2 n^2 t}{L^2}\right) \tag{16}$$

Equations 15 and 16 are the solutions of Eq. 1 with an initial condition in the form of Eq. 3 discussed above (Carslaw and Jaeger 1959). In fact, when d=0 is substituted into Eq. 12, we can obtain Y=Y_0 for all values of X, except for X=0 and X=L. This result is exactly the same condition as described by Eq. 3. It is interesting to find that the difference between the proposed solution (Eq. 13) and the USBR's equation is less than 2%.

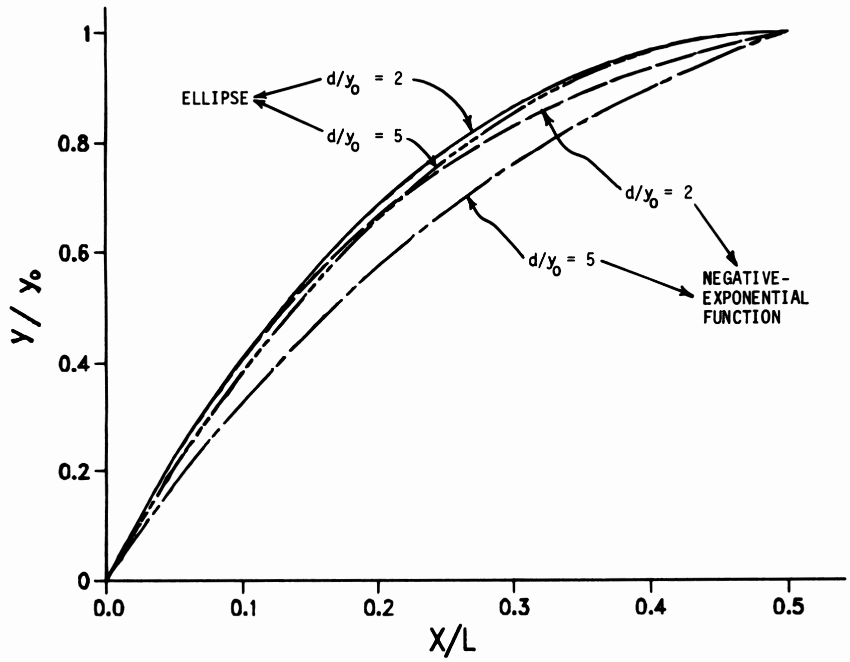

Figure 3. Shapes of ellipse and of negative-exponential functions for different d/Y_0 values.

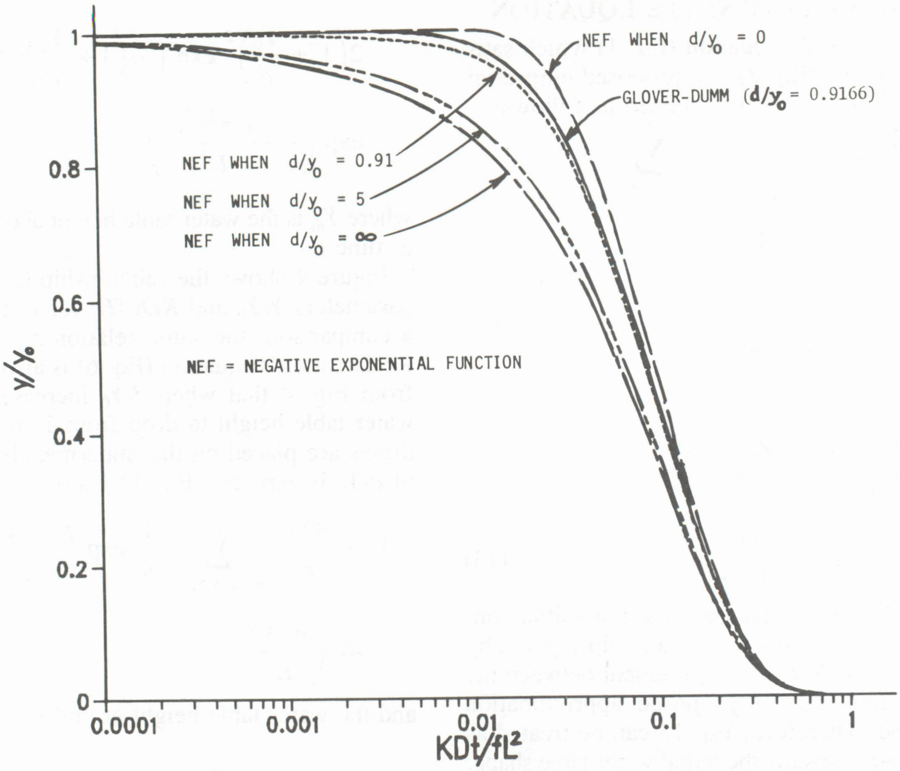

Figure 4. Relationship between Y/Y_0 and KDt/fL^2 for different d/Y_0 values.

NEW DRAIN-SPACING FORMULA

It has been found that the sum of the second and remaining terms in the series, in Eq. 14, is very small and can be neglected. This gives the approximate solution of Eq. 14, by taking only the first term of the series, which yields:

$$Y_m = \frac{4Y_0}{1-\exp[-(1+Y_0/d)^2]}\left\{\frac{1}{\pi} - \frac{1}{4(1+Y_0/d)^4+\pi^2} \cdot \left[\pi - 2\,(Y+Y_0/d)^2 \exp\,(-(1-Y_0/d)^2\right]\right\} \cdot \exp\,(-\alpha\pi^2 t/L^2) \qquad (17)$$

From Eq. 17, a new formula for drain-spacing calculation is developed as follows:

$$L^2 = \frac{\alpha\pi^2 t}{\ln \dfrac{8Y_0\omega\,[2\omega + \pi \exp\,(-\omega)]}{\pi Y_m\,(4\omega^2+\pi^2)\,[1-\exp\,(-\omega)]}} \qquad (18)$$

where $\omega = (1 + Y_0/d)^2$.

In the case of $d = 0$ (i.e., ω approaches infinity) Eq. 18 becomes:

$$L^2 = \frac{\alpha\pi^2 t}{\ln\,(4Y_0/\pi Y_m)} = \frac{\alpha\pi^2 t}{\ln\,(1.27Y_0/Y_m)} \qquad (19)$$

When d approaches infinity, the value of ω approached unity and Eq. 18 becomes:

$$L^2 = \frac{\alpha\pi^2 t}{\ln \dfrac{8Y_0[2 + \pi \exp\,(-1)]}{\pi Y_m\,(4+\pi^2)\,(1-\exp\,(-1))}} = \frac{\alpha\pi^2 t}{\ln\,(0.91Y_0/Y_m)} \qquad (20)$$

It should be mentioned that Eq. 19 is exactly the same as the well-known Glover's drain-spacing equation (Van Schilfgaarde 1974).

It was interesting to find that when the ratio of $d/Y_0 = 0.9100$, is used in Eq. 18, the following equation is obtained:

$$L^2 = \frac{\alpha\pi^2 t}{\ln\,(1.15\,\dfrac{Y_0}{Y_m})} \qquad (21)$$

Equation 21 is almost the same as the Glover-Dumm formula reported by Dumm (1960). The only difference between Eq. 21 and Glover-Dumm's equation is the ''constant'' term in the equation (i.e., 1.15 in Eq. 21 and 1.16 in Glover-Dumm). The difference between these two equations is found to be less than 1%.

CONCLUSION

A new solution of the differential equation (Eq. 13) describing the transient-state drainage problem was presented. The ellipse initial condition was used and was approximated by a negative exponential function in the form of Eq. 12. Good agreement between the negative exponential function and the true ellipse was found. Owing to this good agreement, the solution obtained with this negative exponential function as an initial condition could be considered as the solution with a true ellipse as the initial condition.

It was found that when the ellipse was used as the initial condition, its shape was influenced by the ratio of the semi-major and semi-minor axes. For this reason, the shape of the initial water table will depend not only on water table height at mid-spacing but also on the distance to the impermeable layer below drains. These features are also represented by the negative exponential function.

The new drain-spacing formula (Eq. 18) proposed in this paper can be considered as a general formula for drain-spacing calculations. It was found that two well-known drain-spacing formulae, Glover's equation (Van Schilfgaarde 1974) and Glover-Dumm's (Dumm 1960, 1964) equation, are covered by this new formula. They can be considered as the solution of two special cases of the proposed formula as given in Eqs. 19 and 21, respectively.

Currently, a nonsteady-state drain-spacing equation based on elliptical initial conditions is not available. Drain spacings calculated from steady and nonsteady-state equations cannot be meaningfully compared. This need was successfully addressed in this study by using the ellipse-initial condition in nonsteady-state flow and developing a new drain-spacing equation.

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