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MATHEMATICAL MODELS FOR FLOOD MANAGEMENT: EFFICIENCY AND RELIABILITY

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ABSTRACT Floods are among the most damaging of natural hazards, and are likely to become more predominant in the future due to the effects of the impacts of climate change. A good understanding of a flood event can only be achieved by means of 1D or 2D hydrodynamic models based on the De Saint Venant (SV) or Shallow Water (SW) equations. The topographic data resolution strongly affects the flood model efficiency. Sound Digital Elevation Models (DEMs) must provide an accurate description of the watershed micro-topography to create a computational mesh in which all the elements that affect flood dynamics and flood propagation are included. Comparisons between first-order (Lax-Friedrichs) and second-order (Lax-Wendroff, MacCormack) schemes show that only the last one provides a reliable picture of the flood wave characteristics, such as shape, length, celerity, peak levels and maximum discharge. The much faster first-order solvers can be used to analyze a great variety of scenarios, letting a deeper investigation of the most hazardous environments to the more accurate second-order algorithms, for the design of efficient defence structural and non-structural measures.

Keywords: Flood control, Digital Elevation Models, Mathematical models.

INTRODUCTION One-third of the annual natural disasters and economic losses, and more than half of the respective victims are flood related. These hazards are likely to become more frequent and more relevant in the future, due to the effects of increase in population, urbanization, land subsidence and the impacts of climate change.

Knowledge and advanced scientific tools play a role of paramount importance in the strain of coping with flooding problems. In this context, flood modelling represents the basis for effective flood mitigation.

The modelling approach aims to provide the best means for assessing and, subsequently, reducing the vulnerability of rural and urban flood prone areas.

By using models, an attempt is made to replace trial and error based strategies, as practised in the past, with more physically-based measures of flood management and control. Mathematical models are the best tools, nowadays available, for the design of efficient flood protection strategies and excellent supporters of decision-makers.

Floods are natural processes that change abruptly in time and space and, therefore, their description requires a fully three-dimensional (3D) approach. Nonetheless, in most practical cases, two-dimensional (2D) models can provide an accurate analysis of a flood event.

To this end, the De Saint Venant (SV) equations, known also as Shallow Water (SW) equations, are solved by using finite difference methods (Heniche et al., 2000), finite volume solvers (Brufau et al., 2004), or finite elements (Gracia et al., 2006).

When risk assessment is required, a large number of scenarios has to be simulated. For instance, in the case of a pensile river, the location of the embankment breakage has to be identified. Unfortunately, as the position of the possible breakage is unknown, there is the need to simulate all the possible alternatives, which might be many, and then select the most and/or the less dangerous to better investigate these scenarios.

As 2D models are time-consuming approaches, the use of fast solvers, in the first phases of the risk assessment process, could be very convenient. In this way a large amount of scenarios can be analysed, to select those that require a more detailed investigation by means of more accurate methods.

In the paper three finite difference solvers have been selected, implemented and applied to a case study to check their strength and wickedness (velocity and accuracy), and their applicability to the different phase of the risk assessment process.

Suggestions for further researches are also provided.

GOVERNING EQUATIONS The 2D approach for unsteady incompressible flow is governed by the De Saint Venant equations.

This set of hyperbolic partial differential equation describes a system of conservation laws with source term (*S*) and can be written in a compact vector form as (Lax and Wendroff, 1960):

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = S \tag{1}$$

where:

$$U = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix} \qquad E = \begin{pmatrix} uh \\ u^2 h + \frac{1}{2} g h^2 \\ uvh \end{pmatrix}$$

$$F = \begin{pmatrix} vh \\ uvh \\ v^2 h + \frac{1}{2} g h^2 \end{pmatrix} \qquad S = \begin{pmatrix} 0 \\ gh(S_{0_x} - S_{f\,x}) \\ gh(S_{0_y} - S_{f\,y}) \end{pmatrix} \tag{2}$$

with t: temporal coordinate; h: water depth; u and v: water velocities in the x and y axis respectively; g: acceleration due to gravity; S_0: bed slope; $S_{f\,x}$ and $S_{f\,y}$: bed resistance terms (or friction slopes) in the x and y axis respectively.

NUMERICAL SCHEMES In the research, three explicit finite difference numerical schemes have been applied.

The first tested solver was the well-known first order Lax-Friedrichs scheme (Lax, 1954), which can be written as follows:

$$\begin{aligned} U_{i,j}^{n+1} &= \frac{1}{4}\left(U_{i-1,j}^{n} + U_{i+1,j}^{n} + U_{i,j-1}^{n} + U_{i,j+1}^{n}\right) - \frac{\Delta t}{2\Delta x}\cdot\left(E_{i+1,j}^{n} - E_{i-1,j}^{n}\right) + \\ &\quad - \frac{\Delta t}{2\Delta y}\cdot\left(F_{i,j+1}^{n} - F_{i,j-1}^{n}\right) + \Delta t \cdot S_{i,j}^{n} \end{aligned} \tag{3}$$

To obtain a higher resolution of shocks and rarefactions fans, very common in flood events, higher-order finite difference schemes can be employed. If downwind differences are used to approximate $U_{i,j}^{n+1}$, this will results in the second-order Lax-Wendroff scheme, modified by Richtmyer (1962). This solver can be seen as a combination of the Lax-Friedrichs (in the first step) and Leap-Frog (in the second step) schemes.

The scheme can be written as:

- First step:

$$\begin{aligned} \hat{U}_{i,j}^{n+1} &= \frac{1}{4}\cdot\left(U_{i+1,j}^{n} + U_{i-1,j}^{n} + U_{i,j+1}^{n} + U_{i,j-1}^{n}\right) - \frac{\Delta t}{2\Delta x}\cdot\left(E_{i+1,j}^{n} - E_{i-1,j}^{n}\right) + \\ &\quad - \frac{\Delta t}{2\Delta y}\cdot\left(F_{i,j+1}^{n} - F_{i,j-1}^{n}\right) + \Delta t \cdot S_{i,j}^{n} \end{aligned} \tag{4}$$

- Second step:

$$U_{i,j}^{n+2} = U_{i,j}^{n} - \frac{\Delta t}{\Delta x}\cdot\left(\hat{E}_{i+1,j}^{n+1} - \hat{E}_{i-1,j}^{n+1}\right) - \frac{\Delta t}{\Delta y}\cdot\left(\hat{F}_{i,j+1}^{n+1} - \hat{F}_{i,j-1}^{n+1}\right) + \Delta t \cdot \hat{S}_{i,j}^{n+1} \tag{5}$$

The third method was the MacCormack scheme. This solver incorporates forward (predictor) and backward (corrector) steps, that can be written as:

- Predictor:

$$\mathrm{U}_{i,j}^{p} = \mathrm{U}_{i,j}^{n} - \tau_x \cdot \nabla_x \mathrm{E}_{i,j}^{n} - \tau_y \cdot \nabla_y \mathrm{F}_{i,j}^{n} + \Delta t \cdot \mathrm{S}_{i,j}^{n} \tag{6}$$

valid for $2 \le i \le N$ and $2 \le j \le M$

- Corrector:

$$\mathrm{U}_{i,j}^{c} = \mathrm{U}_{i,j}^{n} - \tau_x \cdot \Delta_x \mathrm{E}_{i,j}^{p} - \tau_y \cdot \Delta_y \mathrm{F}_{i,j}^{p} + \Delta t \cdot \mathrm{S}_{i,j}^{p} \qquad (7)$$

valid for $1 \le i \le N-1$ and $1 \le j \le M-1$. In Eq. 6 the operator ∇ is the backward difference while in Eq. 7 the operator Δ is the forward difference.

In all cases the depth *h* is computed directly being the first row of the vector *U*, while the velocities *u* and *v* in the two directions *x* and *y* can be computed from the second and third rows of the vector *U* as:

$$u_{i,j}^{n+1} = \frac{(uh)_{i,j}^{n+1}}{h_{i,j}^{n+1}} \qquad (8)$$

$$v_{i,j}^{n+1} = \frac{(vh)_{i,j}^{n+1}}{h_{i,j}^{n+1}} \qquad (9)$$

In general, methods based on finite difference approximations may behave well for smooth solutions, but can give spurious oscillations when discontinuities are present. To eliminate these oscillations some forms of dissipation or limiter are used. Oscillations in the second-order MacCormack scheme can be reduced by using artificial viscosity to the conserved variables at the predictor corrector steps, which can be written as (Jameson, 1982):

$$d_x = \varepsilon_x^{(2)} \Delta x \frac{\partial U}{\partial x} - \varepsilon_x^{(4)} \Delta x^3 \frac{\partial^3 U}{\partial x^3} \qquad (10)$$

$$\varepsilon_x^{(2)} = \alpha^{(2)} \left(|u| + c \right) \Delta x^2 \frac{1}{p} \left| \frac{\partial^2 p}{\partial x^2} \right| \qquad (11)$$

where *p* is the pressure, *u* and $c = \sqrt{gh}$ the wave velocity and celerity respectively, while $\varepsilon_x^{(2)}$ and $\varepsilon_x^{(4)}$are the terms that contain the calibration parameters $\alpha^{(2)}$ and $\alpha^{(4)}$.

In the discrete form the dissipative term d_x can be written as follows:

$$d_x = d_{i+\frac{1}{2},j} - d_{i-\frac{1}{2},j} \qquad (12)$$

where:

$$d_{i+\frac{1}{2},j} = \varepsilon_{i+\frac{1}{2},j}^{(2)} \left(U_{i+1,j} - U_{i,j} \right) - \varepsilon_{i+\frac{1}{2},j}^{(4)} \left(U_{i+2,j} - 3U_{i+1,j} + 3U_{i,j} - U_{i-1,j} \right) \qquad (13)$$

$$d_{i-\frac{1}{2},j} = \varepsilon_{i-\frac{1}{2},j}^{(2)} \left(U_{i,j} - U_{i-1,j} \right) - \varepsilon_{i-\frac{1}{2},j}^{(4)} \left(U_{i+1,j} - 3U_{i,j} + 3U_{i-1,j} - U_{i-2,j} \right) \qquad (14)$$

$$\varepsilon^{(2)}_{i+\frac{1}{2},j} = \max\left(\varepsilon^{(2)}_{i,j}, \varepsilon^{(2)}_{i+1,j}\right)$$

$$\varepsilon^{(2)}_{i,j} = \alpha^{(2)} \frac{\left|h_{i+1,j} - 2h_{i,j} + h_{i-1,j}\right|}{h_{i+1,j} + 2h_{i,j} + h_{i-1,j}} \tag{15}$$

$$\varepsilon^{(4)}_{i+\frac{1}{2},j} = \max\left(0, \alpha^{(4)} - \varepsilon^{(2)}_{i+\frac{1}{2},j}\right)$$

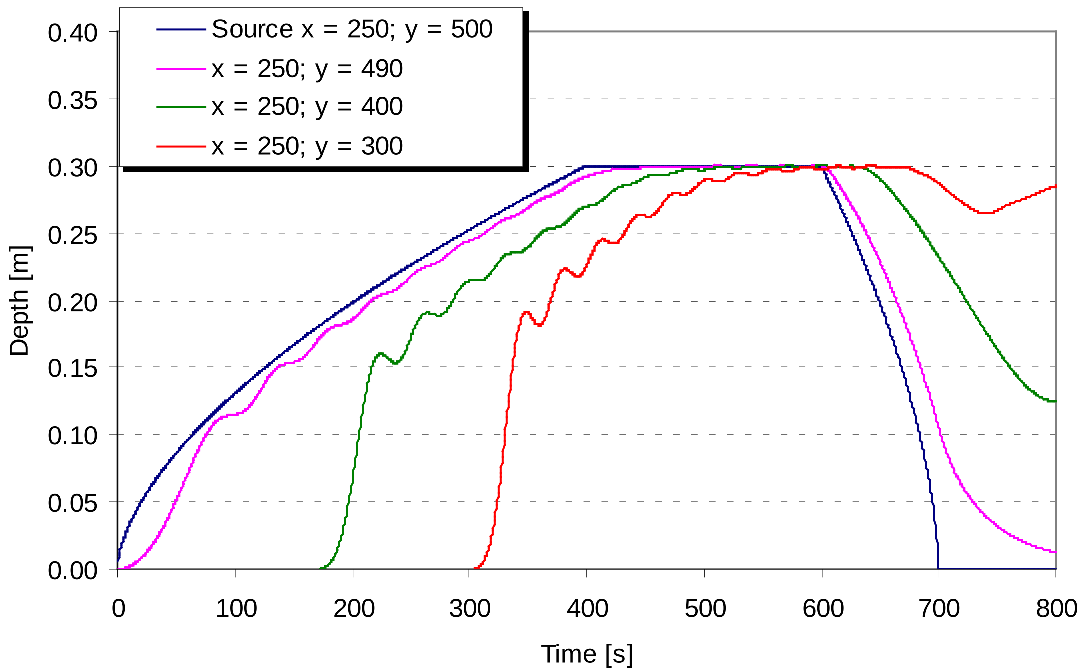
RESULTS. To validate the models comparisons have been made among the simulations provided by the three numerical solvers regarding a plane (51 x 50 squared cells), with side equal to 10 *m*, slope of 5 ‰ and a Strickler's roughness coefficient equal to 35 $m^{1/3}/s$ and a source at the top of the plane, reproducing flood waves. The time step for the simulation of the waves have been chosen according to the solvers' requirements.

In general terms, comparisons among simulations demonstrated that the MacCormack scheme always requires the artificial viscosity terms, while the Lax-Friedrichs and Lax-Wendroff solvers are sufficiently stable without limiters.

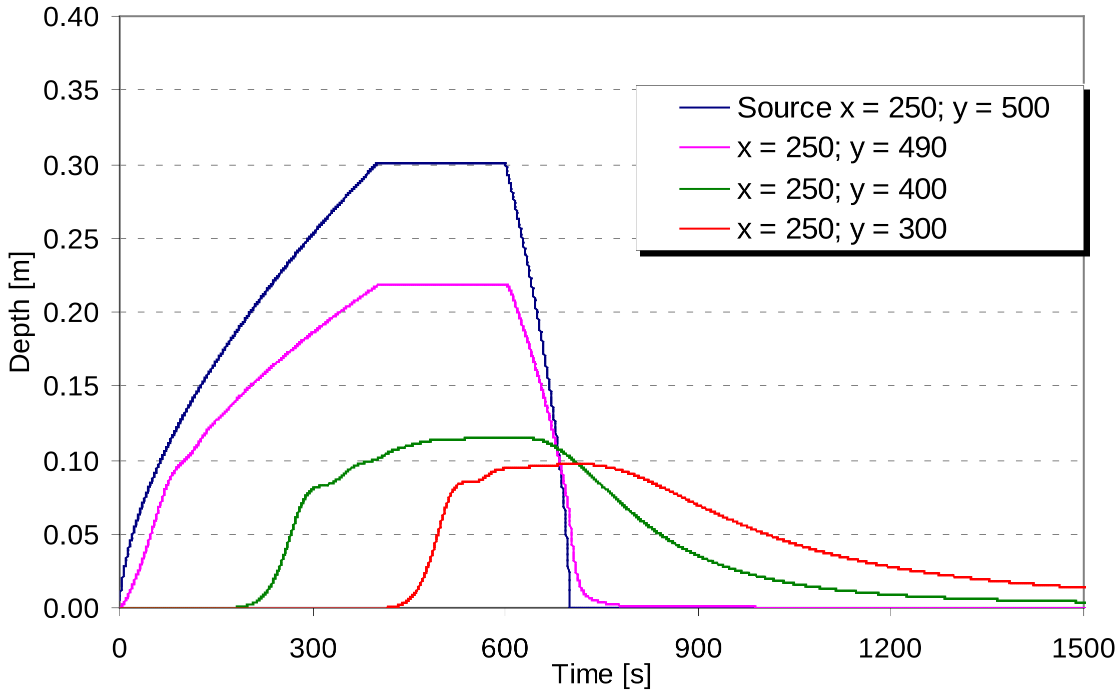
Figure 1 gives the time evolution of water depth at some locations downstream of the source using MacCormack solver (a) without and (b) with artificial viscosity. Without dissipative terms and due to the strong oscillations the code (implemented by the Authors) fails to provide feasible solutions and overflows before ending.

Comparisons among 2D simulations of the flood wave at time *t*=10 *s*, are shown in figure 2, computed with $\Delta t = 0.005\ s$. Only the MacCormack scheme provides feasible results, while both Lax-Friedrichs and Lax-Wendroff solvers give over-predicted solutions. As a matter of fact, the flood wave could not have reached distances more than 150 *m* downstream in only 10 *s*, with velocity too high to be acceptable. These schemes seem to have difficulties to accurately predict wave velocity for dry bed or planes downstream of the source. This effect is due to numerical dispersion, which can be assessed by means of the Relative Phase Error (RPE) factor (LeVeque, 2002; Quarteroni, 2003). According to this parameter, numerical waves are faster than real for RPE > 1, and slower for RPE<1.

With regard to the Lax-Friedrichs scheme, it has been found that the best phase and amplitude accuracy are obtained for Courant-Friedrichs-Lewy (CFL) numbers close to one, while for the MacCormack solver only very small time steps are appropriate to optimize the accuracy.

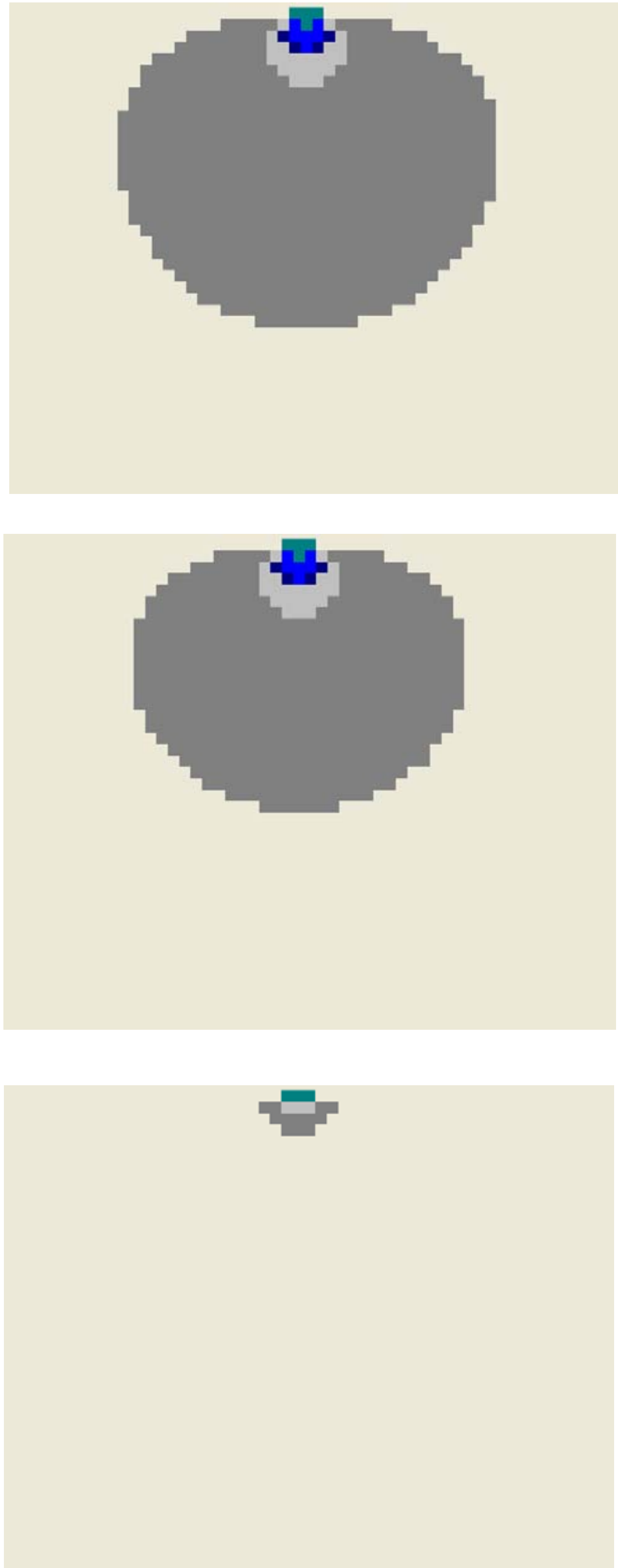


(1.a)



(1.b)

Figure 1 – Time evolution of water depth at some locations downstream of the source, using MacCormack second-order scheme, (1.a) without and (1.b) with artificial viscosity.



(2.a)

(2.b)

(2.c)

Figure 2 – 2D simulation of water depth at time *t*=10 *s* (Δ*t*=0,005 *s*) using (2.a) Lax-Friedrichs first-order scheme, (2.b) Lax- Wendroff second-order scheme, (2.c) MacCormack second-order scheme, with artificial viscosity.

	Lax-Friedrichs	Lax-Wendroff	MacCormack	
Δt (*s*)	0.01	0.01	0.01	0.005
Duration (*s*)	1800	1800	1800	1800
Time for computation	06'45''	10'54"	Overflow	27'16"
CFL	0.00354	0.00354	0.00354	0.00182

Table 1 – Comparisons of the performances of Lax-Friedrichs first-order and Lax-Wendroff second-order schemes.

Δt (*s*)	0.01	0.5	2
Duration (*s*)	1800	1800	1800
Time for computation	06'45"	00'10"	00'03"
CFL	0.00354	0.177	0.707

Table 2 – Simulation characteristics using Lax-Friedrichs first-order scheme.

The performances of the Lax-Friedrichs, Lax-Wendroff and MacCormack schemes are reported in Table 1, while in Table 2 the time characteristics (step, duration and computational times) and CFL numbers for the Lax- Friedrichs scheme are given.

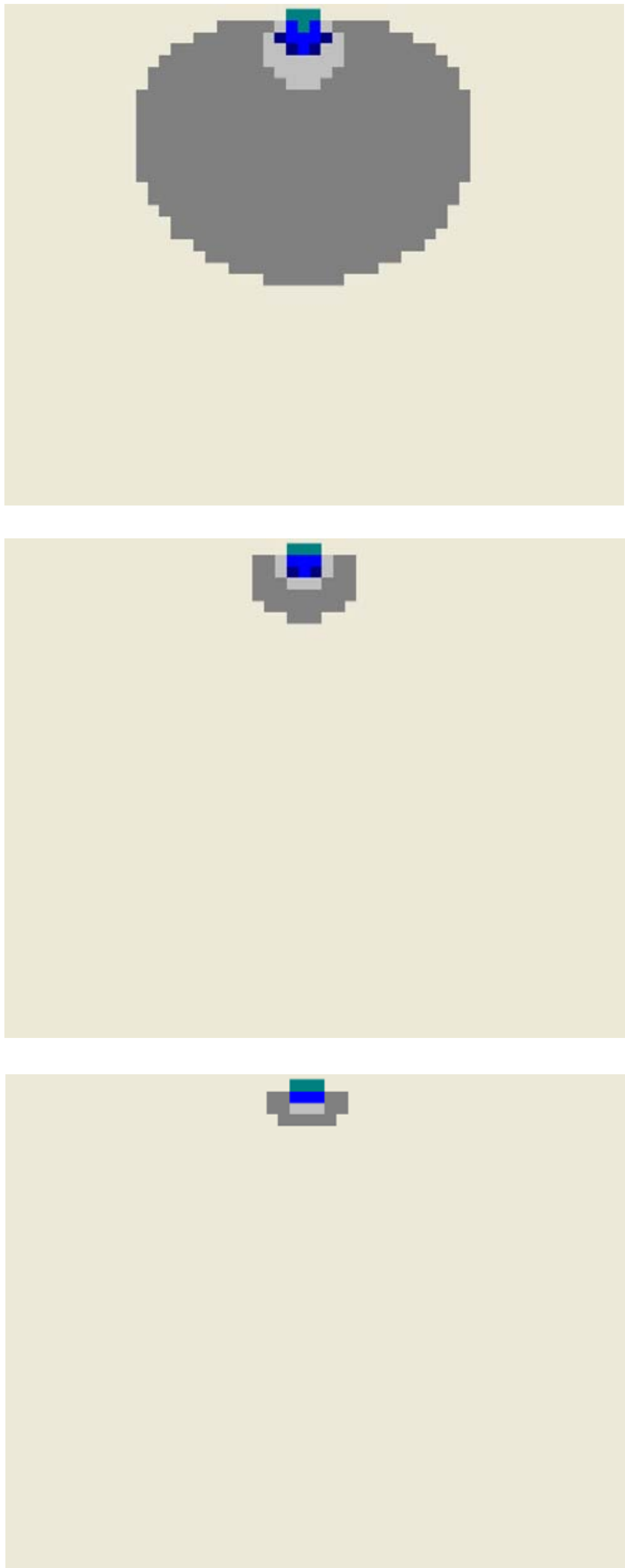
Figure 3 shows the improvements that can be obtained by increasing the temporal steps. As a matter of fact, increasing Δt not only dramatically reduces the computational time, but also greatly increases the accuracy of the results.

CONCLUSION Floods are among the most damaging of natural hazards, and are likely to become more relevant in the future due to the effects of the impacts of climate changes. A good understanding of a flood event can only be achieved by means of hydrodynamic models based on the SV or SW equations. The modeling approach aims to provide the best concept for assessing and reducing the vulnerability of rural and high-value urban flood-prone areas as well as industrial zones.

Within this ground, a number of numerical schemes for solving the 2D SV equations, applied to simulate flood scenarios, have been examined. These include the first-order Lax-Friedrichs and the second-order Lax-Wendroff and MacCormack schemes.

With the exception of second-order Lax-Wendroff solver, very time consuming and inaccurate, all other schemes produce reasonable results and can be successfully used in the practice engineering.

On the whole, Lax-Friedrichs scheme is less accurate with respect to the MacCormack solver, which is second-order accurate, both in the time and space .Therefore this scheme could be used to investigate a large amount of scenarios, to select those that require a more detailed analysis, which can be carried out using the MacCormack solver, for the design of efficient structural and non-structural flood defense measures.



(3.a)

(3.b)

(3.c)

Figure 3 – 2D simulation of water depth at time t=10 s (Δt=0,005 s) using Lax-Friedrichs first-order scheme with time step (3.a) $\Delta t = 0.01\, s$, (3.b) $\Delta t = 0.5\, s$, (3.c) $\Delta t = 2.0\, s$

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